

Test 1

MA 464 (Theory of Probability)

October 5, 2017

Name: \_\_\_\_\_

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**SHOW ALL YOUR WORK!**

1. [20 points] Diseases  $D_1$  and  $D_2$  cause symptom A with probabilities 0.4 and 0.8, respectively. Suppose 6% of a population have disease  $D_1$  and 2% have disease  $D_2$ . Assume that the only possible causes of symptom A are  $D_1$  and  $D_2$ , and that no one carries more than one of those two diseases.
  - (a) What percent of the population have symptom A?
  - (b) Let a randomly selected person from the population have symptom A. What is the chance he/she carries disease  $D_1$ ?

2. [20 points] A discrete random variable  $X$  has probability mass function  $f_X$  defines as follow

$$f_X(x) = \begin{cases} c(x^2 + 1), & \text{if } x = 0, \pm 1, \pm 2. \\ 0 & \text{otherwise} \end{cases}$$

where  $c$  is a constant. Find the following

- (a) the value of  $c$ .
- (b)  $\mathbb{P}(-1 \leq X < 2)$ .
- (c)  $\mathbb{P}(|X| \leq 1 \mid X \geq 0)$ .
- (d)  $\mathbb{E}(2X - 1)$

3. [20 points] Suppose  $X$  and  $Y$  are two independent random variables such that

$$\mathbb{E}(X) = 2, \text{Var}(X) = 4, \quad \mathbb{E}(Y) = -7, \text{Var}(Y) = 9.$$

Find the following

- (a)  $\mathbb{E}(X^2)$  and  $\mathbb{E}(Y^2)$ .
- (b)  $\mathbb{E}(2X^2 - Y^2 - 3XY)$ .
- (c)  $\mathbb{E}(XY(X - Y))$ .
- (d)  $\text{Var}(XY)$ .

4. [20 points] The probability of triplets in human birth is approximately 0.001. Use poisson approximation to find the probability that, among 700 births in a large hospital, there will be
- (a) at least one set of triplets?
  - (b) at most one set of triplets?

5. [20 points] You have three alarm clocks that will ring on any given morning with probabilities 0.8, 0.85, and 0.9, respectively. To wake up on an important exam day, you set all the three alarm clocks.

- (a) What is the probability that exactly two alarm clocks will ring?
- (b) What is the probability of you being awakened by your alarm clocks?

6. [15 points] Let  $X$  be a discrete random variable with a geometric distribution whose mean is 2.
- (a) Find  $\mathbb{P}(X \geq 6 \mid X > 4)$ , your answer should not have a summation.
  - (b) Let  $Z = -X + 3$ . Find the mean and variance of  $Z$ .
  - (c) Compute the conditional expectation of  $X$  given that  $X$  is at least 3.

In the following  $0 < p < 1$  and  $q := 1 - p$ .

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**Bernoulli:**  $X \sim \text{Bernoulli}(p)$

$$\begin{aligned} \mathbb{P}(X = 1) &= p \quad \text{and} \quad \mathbb{P}(X = 0) = q. \\ \mathbb{E}(X) &= p, \quad \text{Var}(X) = pq, \quad M_X(t) = q + pe^t. \end{aligned}$$

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**Binomial:**  $X \sim \text{Binomial}(n, p)$ ,  $n \in \mathbb{N}$ .

$$\begin{aligned} \mathbb{P}(X = k) &= \binom{n}{k} q^{n-k} p^k, \quad k = 0, 1, \dots, n. \\ \mathbb{E}(X) &= np, \quad \text{Var}(X) = npq, \quad M_X(t) = (q + pe^t)^n. \end{aligned}$$

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**Geometric:**  $X \sim \text{Geometric}(p)$

$$\begin{aligned} \mathbb{P}(X = k) &= q^{k-1} p, \quad k = 1, 2, \dots \\ \mathbb{E}(X) &= \frac{1}{p}, \quad \text{Var}(X) = \frac{q}{p^2}, \quad M_X(t) = \frac{pe^t}{1 - qe^t}. \end{aligned}$$

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**Poisson:**  $X \sim \text{Poisson}(\lambda)$ ,  $\lambda > 0$ .

$$\begin{aligned} \mathbb{P}(X = k) &= \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, \dots \\ \mathbb{E}(X) &= \lambda, \quad \text{Var}(X) = \lambda, \quad M_X(t) = e^{\lambda(e^t - 1)}. \end{aligned}$$

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**Negative Binomial:**  $X \sim \text{NB}(n, p)$ ,  $n \in \mathbb{N}$ .

$$\begin{aligned} \mathbb{P}(X = k) &= \binom{k-1}{n-1} p^n q^{k-n}, \quad k = n, n+1, \dots \\ \mathbb{E}(X) &= \frac{n}{p}, \quad \text{Var}(X) = \frac{nq}{p^2}, \quad M_X(t) = \left( \frac{pe^t}{1 - qe^t} \right)^n. \end{aligned}$$


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